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The effect of normal blowing on the flow near a rotating disk of infinite extent

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The effect of blowing through a porous rotating disk on the flow induced by this disk is studied. For strong blowing the flow is almost wholly inviscid. First-order viscous effects are encountered only in a thin layer at some distance from the disk. The results of an asymptotic analysis are compared with numerical integrations of the full equations and complete agreement is found.

1. Introduction

Ever since von Kármán (1921) derived the simplified equations that govern the flow over an infinite rotating disk this problem, and many variations of it, have attracted many authors. An excellent source of reference, covering most of the work done before 1960, is a book by Dorfman (1963). The effect of suction, which was studied first by Stuart (1954), later received the attention of Rogers & Lance (1960) and Evans (1969). These theoretical studies cover the range from weak to strong suction. Gregory & Walker (1960) performed experiments on this subject.

The influence of blowing has been investigated by Sparrow & Gregg (1960) who included suction as well. These authors consider a rotating porous disk, in an infinite expanse of fluid, through which additional fluid is injected normally and uniformly into the system. The governing ordinary differential equations were integrated numerically for weak, intermediate, and moderately strong blowing.

In the present paper we reconsider this problem, but we shall be especially concerned with strong blowing because of its interesting nature. In other areas of fluid mechanics the study of strong blowing has recently come into prominence. Acrivos (1962) considered large mass transfer from an ablating surface, which results in the viscous boundary layer being blown away from the surface. Watson (1966) investigated large injection rates that give rise to similar boundary layers. From these studies it is clear that in the limiting case of infinitely strong blowing, the injected fluid flow is inviscid. This layer is separated from the outer flow, which is also inviscid, by a vortex sheet. For finite injection rates this will be a viscous boundary layer.

2. Equations and boundary conditions

We use cylindrical polar co-ordinates (r, ϕ, z) and denote the corresponding velocity components by (u, v, w). The plane $z = 0$ rotates about the z-axis with constant angular velocity Ω and the injection rate of fluid is w_0. Thus at $z = 0$ we have $u = 0$, $v = \Omega r$, $w = w_0 > 0$. The equations to be used are the equation of continuity and the r, ϕ and z components of the Navier–Stokes equation for incompressible fluids. Upon solving the equation of continuity in the familiar way by putting

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = \frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{1}$$

we substitute

$$\psi = \tfrac{1}{2}w_0 r^2 f(\eta); \quad v = \Omega r g(\eta); \quad \overline{p}/\overline{\rho} = w_0^2 p(\eta); \quad \eta = (\Omega/w_0)z, \tag{2}$$

where $\overline{p}$ is the pressure and $\overline{\rho}$ the density, in the three remaining equations and obtain

$$ff'' - \tfrac{1}{2}(f')^2 + 2g^2 = Rf''', \tag{3}$$

$$fg' - gf' = Rg'', \tag{4}$$

$$ff' + p' = Rf''. \tag{5}$$

Primes stand for differentiation with respect to the argument. The boundary conditions at the disk, which were described above, are now given by

$$f(0) = 1; \quad f'(0) = 0; \quad g(0) = 1. \tag{6}$$

Far above the disk u, v and p must vanish, so that

$$f'(\infty) = 0; \quad g(\infty) = 0; \quad p(\infty) = 0. \tag{7}$$

It should be remarked that here the pressure is the difference between the actual and the hydrostatic pressure. Just as in von Kármán's problem the axial velocity component w will not vanish for $z \to \infty$, since there must be an axial inflow to feed the radial outflow along the disk.

From (3)–(7) it can be seen that the only parameter left in the equations is

$$R = \nu\Omega/w_0^2. \tag{8}$$

It follows that for very strong blowing (w_0 large) this parameter can be small. It can also become small for a given blowing rate if the disk is made to rotate sufficiently slowly. Also the inviscid limit ($\nu \to 0$) is represented by $R \to 0$.

3. Injection layer

By expanding in the equations (3)–(5) using series expansions of the type

$$f = f_0 + Rf_1 + R^2 f_2 + \ldots \tag{9}$$

and similarly for g and p, we obtain a solution for the flow in the injection layer. This has been shown by Watson (1966) for an analogous problem. As the highest derivatives are neglected in the first-order solution, we clearly consider the inviscid limit in this way. As boundary conditions we must consider only those given at the disk.

After solving the requisite equations the following solutions are obtained

$$f_0 = \cos^2\eta, \tag{10}$$

$$g_0 = \cos^2\eta, \tag{11}$$

$$f_1 = -\tfrac{16}{3}\sin\eta\cos\eta\ln(\cos\eta), \tag{12}$$

$$g_1 = \tfrac{2}{3}\tan\eta - \tfrac{8}{3}\sin\eta\cos\eta - \tfrac{16}{3}\sin\eta\cos\eta\ln(\cos\eta), \tag{13}$$

$$f_2 = \tfrac{68}{9}\eta\sin\eta\cos\eta - \tfrac{16}{27}\cos^{-2}\eta + \tfrac{44}{27} - \tfrac{28}{27}\cos^2\eta + \tfrac{128}{9}\sin^2\eta\ln(\cos\eta) + \tfrac{64}{9}(1-2\cos^2\eta)\ln^2(\cos\eta), \tag{14}$$

$$g_2 = \tfrac{68}{9}\eta\sin\eta\cos\eta + \tfrac{2}{9}\cos^{-4}\eta - \tfrac{16}{27}\cos^{-2}\eta + \tfrac{332}{27} - \tfrac{322}{27}\cos^2\eta + \tfrac{16}{9}(\cos^{-2}\eta + 12 - 16\cos^2\eta)\ln(\cos\eta) + \tfrac{64}{9}(1-2\cos^2\eta)\ln^2(\cos\eta), \text{ etc.} \tag{15}$$

As this solution is developed using all boundary conditions prescribed at the wall, it can be expected that it is valid in a region near the wall. For $R = 0$ we have $f = g = \cos^2\eta$. This means that the outer boundary of the region, where the present expansion is valid, is given by $\eta = \frac{1}{2}\pi$. Indeed, the zero streamline, i.e. the one issuing from the centre of rotation ($r = 0, z = 0$), is given by $f = 0$.

The true meaning of the zero-order term, i.e. the inviscid flow, becomes clear if we consider the paths traversed by the fluid particles. If we focus our attention on a particle launched at $(r_0, \phi_0, 0)$ we obtain from (1), (2), (10) and (11)

$$d\eta/d\tau = \cos^2\eta, \tag{16}$$

$$d\rho/d\tau = \rho\sin\eta\cos\eta, \tag{17}$$

$$d\theta/d\tau = \cos^2\eta, \tag{18}$$

where $\rho = r/r_0$, $\tau = \Omega t$ and $\theta = \phi - \phi_0$. Using $\eta = 0$, $\rho = 1$, $\theta = 0$ at $\tau = 0$, we obtain

$$\eta = \arctan\tau; \quad \rho = (1+\tau^2)^{\frac{1}{2}}; \quad \theta = \arctan\tau, \tag{19}$$

so that $\rho\cos\theta = 1$. Thus the projection of the path on the plane of the disk is a straight line through the point of injection, the direction being determined by the direction of v at the moment of injection. All particles approach the plane $\eta = \frac{1}{2}\pi$ as τ tends to infinity.

Another interesting feature is the velocity of the particle. After a simple calculation we obtain

$$|v_p| = \Omega r_0\left\{1 + \frac{w_0^2}{\Omega^2 r_0^2}(1+\tau^2)^{-2}\right\}^{\frac{1}{2}}, \tag{20}$$

which gives the expected value at $\tau = 0$. As time progresses $|v_p|$ will decrease continually and finally it will attain a value which is equal to the azimuthal component of the velocity of the particle at the moment of injection. It is easy to show that the projection of $\mathbf{v}_p$ on the disk is equal to Ωr_0, independently of time.

From the analytical expression for the higher perturbations it is evident that at $\eta = \frac{1}{2}\pi$ the expansion loses its meaning. At $\eta = \frac{1}{2}\pi$ the higher perturbations display singularities of increasing strength. Thus, no matter how small R is, we can always choose η so close to $\frac{1}{2}\pi$ that the series diverges right from the start.

Therefore, no proper matching with the flow beyond the plane $\eta = \frac{1}{2}\pi$ is possible. Acrivos (1962) and Watson (1966) encountered the same difficulty. These authors then decided that there should be a viscous boundary layer in the immediate neighbourhood of $\eta = \frac{1}{2}\pi$. At the disk side the flow in this sublayer should match with that in the injection layer described above. At the opposite side it should satisfy the conditions posed at $\eta = \infty$. This viscous sublayer will be described in the next section.

4. Viscous sublayer

Upon introduction of

$$f(\eta) = R^{\frac{2}{3}}F(\mu); \quad g(\eta) = R^{\frac{2}{3}}G(\mu); \quad \eta = \tfrac{1}{2}\pi + R^{\frac{1}{3}}\mu, \tag{21}$$

the equations (3) and (4) are transformed into

$$F''' - FF'' + \tfrac{1}{2}(F')^2 = 2R^{\frac{2}{3}}G^2, \tag{22}$$

$$G'' - FG' + F'G = 0. \tag{23}$$

It is seen that the viscous terms are no longer small. Moreover, the second derivatives of the outer functions (f and g) are of the same order of magnitude as those of the corresponding inner functions (F and G), if R tends to zero, which is required for proper matching. Indeed, both f and g are $\sim \cos^2\eta$ for small R, so that these functions and their first derivatives are approximately equal to zero near $\eta = \frac{1}{2}\pi$. The second derivatives, however, are of order unity. It is easy to show that under these conditions the transformation (21) is unique.

From (7) we derive the boundary conditions to be satisfied by F and G as μ tends to infinity

$$F'(\infty) = 0; \quad G(\infty) = 0. \tag{24}$$

The matching conditions for $\mu \to -\infty$ are obtained from

$$\begin{aligned}
\lim_{\substack{\mu\to-\infty \\ R\to 0}} R^{\frac{2}{3}}F(\mu) &= \lim_{\substack{\mu\to-\infty \\ R\to 0 \\ |R^{\frac{1}{3}}\mu| \ll 1}} f(\tfrac{1}{2}\pi + R^{\frac{1}{3}}\mu) \\
&= \mu^2 R^{\frac{2}{3}} + \tfrac{16}{3}\mu R^{\frac{4}{3}} \ln R^{\frac{1}{3}} + \left\{-\frac{\mu^4}{3} + \tfrac{16}{3}\mu \ln(-\mu) - \frac{16}{27}\frac{1}{\mu^2} + \dots\right\} R^{\frac{4}{3}} \\
&\quad + \tfrac{64}{9} R^2 \ln^2 R^{\frac{1}{3}} + \{-\tfrac{32}{9}\mu^3 + \tfrac{128}{9}\ln(-\mu) \\
&\quad + \tfrac{128}{9} + \dots\} R^2 \ln R^{\frac{1}{3}} + O(R^2),
\end{aligned} \tag{25}$$

$$\begin{aligned}
\lim_{\substack{\mu\to-\infty \\ R\to 0}} R^{\frac{2}{3}}G(\mu) &= \lim_{\substack{\mu\to-\infty \\ R\to 0 \\ |R^{\frac{1}{3}}\mu| \ll 1}} g(\tfrac{1}{2}\pi + R^{\frac{1}{3}}\mu) \\
&= \left\{\mu^2 - \frac{2}{3}\frac{1}{\mu} + \frac{2}{9}\frac{1}{\mu^4} + \dots\right\} R^{\frac{2}{3}} + \left\{\tfrac{16}{3}\mu + \frac{16}{9}\frac{1}{\mu^2} + \dots\right\} R^{\frac{4}{3}} \ln R^{\frac{1}{3}} \\
&\quad + \left\{-\frac{\mu^4}{3} + \tfrac{16}{3}\mu \ln(-\mu) + \tfrac{26}{9}\mu + \frac{16}{9}\frac{\ln(-\mu)}{\mu^2}\right. \\
&\quad \left. - \frac{4}{9}\frac{1}{\mu^2} + \dots\right\} R^{\frac{4}{3}} + O(R^2 \ln^2 R^{\frac{1}{3}}).
\end{aligned} \tag{26}$$

Therefore, by writing

$$\begin{aligned}
F(\mu) = F_0(\mu) + F_1(\mu) R^{\frac{2}{3}} \ln R^{\frac{1}{3}} + F_2(\mu) R^{\frac{2}{3}} + F_3(\mu) R^{\frac{4}{3}} \ln^2 R^{\frac{1}{3}} \\
+ F_4(\mu) R^{\frac{4}{3}} \ln R^{\frac{1}{3}} + O(R^{\frac{4}{3}})
\end{aligned} \tag{27}$$

and similarly for $G(\mu)$, matching is possible.

By inserting expansions of the type (27) into (22) and (23) the following zeroth-order equations are obtained

$$F_0''' - F_0 F_0'' + \tfrac{1}{2}(F_0')^2 = 0, \tag{28}$$

$$G_0'' - F_0 G_0' + F_0' G_0 = 0, \tag{29}$$

showing the surprising fact that F_0 is uncoupled from G_0. In other words: in the viscous sublayer the axial and radial motions are uncoupled from the motion in the azimuthal direction, i.e. to first order. Thus, we can solve (28) first.

It can be shown by substitution and collecting terms of highest order that the asymptotic behaviour of F_0 as $\mu \to -\infty$ is given by (α, β and γ general constants)

$$F_0 \sim \alpha(\mu+\beta)^2 + \gamma\alpha^{\frac{1}{3}}(\mu+\beta)\int_{-\infty}^{\alpha^{\frac{1}{3}}(\mu+\beta)} dt \int_{-\infty}^{t} \frac{\exp(s^3/3)}{s^3}\, ds + \ldots. \tag{30}$$

Thus by taking $\alpha = 1$, $\beta = 0$ the matching condition $F_0 \to \mu^2$ as $\mu \to -\infty$ can be satisfied. In fact, this condition is satisfied by a one-parameter set of solutions. By choosing γ conveniently the condition $F_0'(\infty) = 0$ can be fulfilled. This is confirmed by numerical integration.

An important result is the value of μ_0 for which $F(\mu_0) = 0$

$$\mu_0 = -0{\cdot}430187. \tag{31}$$

Indeed, this determines, to first order, the plane that separates the region of ambient fluid from that of injected fluid. Other pertinent figures are given in table 1.

i	$F_i(\mu_0)$	$F_i'(\mu_0)$	$F_i''(\mu_0)$	$F_i(\infty)$	$G_i(\mu_0)$	$G_i'(\mu_0)$
0	0·0	−1·295614	1·223951	−1·209625	0·835209	−0·977468
1	obtainable from $i = 0$					
2	1·325093	2·340304	−4·633694	1·573091	−1·471894	3·100884

TABLE 1

From the fact that F_0 tends to a negative constant as $\mu \to \infty$ we derive immediately using (28)

$$F_0' \sim c \exp(F_0(\infty)\mu) \quad \text{as} \quad \mu \to \infty, \tag{32}$$

where c is a constant to be determined, and thus the exponential decay is guaranteed. Indeed, the flow above the boundary layer is potential and matching of a boundary layer with such a flow should involve only exponentially small errors (Goldstein 1965). It should be noted that the inviscid layer near the plate is not a potential flow and thus the argument of Goldstein does not apply to matching with this layer. We will see in fact that the matching of higher orders will involve terms that decay algebraically.

We will devote our attention now to the integration of (29) using as boundary conditions $G_0(\infty) = 0$ and the matching condition obtainable from (26). This matching condition shows that the behaviour of G_0 for $\mu \to -\infty$ is rather special and this will cause us to have a closer look at the asymptotic behaviour of solutions to (29). If we neglect terms of exponentially small order in the coefficients, the asymptotic behaviour of G_0 for $\mu \to -\infty$ should be obtained from

$$G_0''' - \mu^2 G_0' + 2\mu G_0 = 0. \tag{33}$$

One can easily convince oneself, by substitution, that the general solution of (33) can be expressed in confluent hypergeometric functions

$$G_0 = A_1 M(-\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}\mu^3) + A_2 M(-\tfrac{1}{3}, \tfrac{4}{3}, \tfrac{1}{3}\mu^3)\,\mu, \tag{34}$$

where

$$M(a, b, x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}. \tag{35}$$

The two independent solutions in (34) have, apart from a constant multiplier, the same asymptotic behaviour as $\mu \to -\infty$ (Abramowitz & Stegun 1965). Using (for $x \to \infty$)

$$M(a, b, -x) \sim \frac{\Gamma(b)}{\Gamma(b-a)} x^{-a} \left[\sum_{n=0}^{R-1} \frac{(a)_n (1+a-b)_n}{n!} x^{-n} + O(x^{-R})\right] + \text{exponentially small terms}, \tag{36}$$

we derive

$$G_0 \sim \left\{3^{-\frac{2}{3}} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})} A_1 - 3^{-\frac{1}{3}} \frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{5}{3})} A_2\right\} \left(\mu^2 - \frac{2}{3}\frac{1}{\mu} + \frac{2}{9}\frac{1}{\mu^4} - \frac{40}{81}\frac{1}{\mu^7} + \dots\right). \tag{37}$$

Thus by appropriate choice of the coefficient in (37) the behaviour of (26) is obtained. This provides one relation for the unknown constants A_1 and A_2, leaving one degree of freedom to satisfy $G_0(\infty) = 0$. To achieve this we of course have to integrate the full equation (29) using the computer. We refer again to table 1 for the results.

It may be of interest to note that the asymptotic behaviour (37) was used to advantage in the numerical integration. By extending the expansion up to μ^{-10} the numerical integration needed only be carried up to $\mu \sim -5$. Applying the condition that the integrated function join smoothly with the known asymptotic behaviour at $\mu = -5$, which is known very accurately, only exponentially small errors are made in view of (36). As a check the integration was carried out up to $\mu = -10$ using the calculated initial conditions. This confirmed the algebraic behaviour of the tail given in (37). Some researchers have reported difficulties in integrating numerically boundary-layer equations that behave asymptotically in an algebraic manner. One must integrate over an extended range for proper satisfaction of asymptotic conditions. However, the present analysis shows that one may get around this by making full use of the asymptotic behaviour.

The first perturbations F_1 and G_1 have to satisfy the equations

$$F_1'''' - F_0 F_1'' + F_0' F_1' - F_0'' F_1 = 0, \tag{38}$$

$$G_1''' - F_0 G_1' + F_0' G_1 - G_0' F_1 + G_0 F_1' = 0, \tag{39}$$

$F_1'(\infty) = G_1(\infty) = 0$, and appropriate matching conditions (see (25)–(26)). The solutions are given by

$$F_1 = \tfrac{8}{3} F_0'; \; G_1 = \tfrac{8}{3} G_0'. \tag{40}$$

Next, the equation for F_2 reads

$$F_2'''' - F_0 F_2'' + F_0' F_2' - F_0'' F_2 = 2G_0^2, \tag{41}$$

the solution of which should satisfy $F_2'(\infty) = 0$ and (25). In order to prove that solutions of (41) can satisfy the matching condition, let us differentiate (41) and retain only terms that are not exponentially small as $\mu \to -\infty$. This yields

$$F_2^{\mathrm{iv}} - \mu^2 F_2''' = 4G_0G_0', \tag{42}$$

where G_0 is given by the expression in parentheses of (37). The general solution of (42) is

$$F_2''' = 4e^{\frac{1}{3}\mu^3}\int^{\mu} e^{-\frac{1}{3}s^3} G_0 G_0' ds + ce^{\frac{1}{3}\mu^3}, \tag{43}$$

where μ is supposed to have large negative values. By repeated partial integration the first term on the right-hand side of (43) can be expressed as

$$-4\left[\frac{G_0G_0'}{\mu^2} + \frac{1}{\mu^2}\left(\frac{G_0G_0'}{\mu^2}\right)' + \frac{1}{\mu^2}\left\{\frac{1}{\mu^2}\left(\frac{G_0G_0'}{\mu^2}\right)'\right\}' + \ldots\right], \tag{44}$$

which upon substitution of the asymptotic behaviour of G_0 renders

$$-8\mu - \frac{16}{3}\frac{1}{\mu^2} + \frac{128}{9}\frac{1}{\mu^5} - \frac{6800}{81}\frac{1}{\mu^8} + \ldots \tag{45}$$

By integrating (45) three times the general asymptotic behaviour of F_2 as $\mu \to -\infty$ is obtained as follows

$$F_2 \sim -\frac{\mu^4}{3} + \tfrac{16}{3}\mu\ln(-\mu) - \tfrac{16}{3}\mu - \frac{16}{27}\frac{1}{\mu^2} + \frac{680}{1701}\frac{1}{\mu^5} + \ldots +$$
$$+ A + B\mu + D\mu^2 + C \times \text{terms of exponentially small order,}$$

where A, B, C and D are constants of integration. On choosing $A = 0$, $B = \frac{16}{3}$ and $D = 0$ it follows that the asymptotic behaviour as prescribed by (25) can be satisfied with exponentially small error. Since C is still not determined it follows that a one-parameter set of solutions exists that satisfies (25). From this set we have to choose the solution that also satisfies $F_2'(\infty) = 0$. This analysis is confirmed by numerical integration.

This process may be continued up to any order desired, but the algebraical work becomes increasingly cumbersome. Since it is clear now that the special behaviour prescribed by the outer solution can be satisfied indeed by solutions of the inner equations, there is no need to dwell on this any further. The results for G_2 have been added in order to complete the inner expansion up to $R^{\frac{4}{3}}$.

A comparison of the results of the approximate treatment given in this section with results of an accurate numerical integration is now possible. The quantities that are most easily compared are $f(\infty)$ and η_0, where $f(\eta_0) = 0$. The asymptotic expression for $f(\infty)$ is immediately obtainable from (21), (27) and table 1. That for η_0 calls for a somewhat more involved analysis. We have

$$\begin{aligned} 0 = f(\eta_0) &= R^{\frac{2}{3}}F(\mu_0 + R^{\frac{2}{3}}\ln R^{\frac{1}{3}}\mu_1 + R^{\frac{2}{3}}\mu_2 + \ldots) \\ &= R^{\frac{2}{3}}[F_0(\mu_0) + \{F_1(\mu_0) + \mu_1 F_0'(\mu_0)\} R^{\frac{2}{3}}\ln R^{\frac{1}{3}} + \{F_2(\mu_0) + \mu_2 F_0'(\mu_0)\} R^{\frac{2}{3}} + \ldots]. \end{aligned} \tag{46}$$

Equating all coefficients in (46) to zero gives the values of μ_1, μ_2, etc. Thus

$$\eta_0 = \tfrac{1}{2}\pi + R^{\frac{1}{3}}(-0{\cdot}430187 - \tfrac{8}{3}R^{\frac{2}{3}}\ln R^{\frac{1}{3}} + 1{\cdot}022753R^{\frac{2}{3}} + \ldots). \tag{47}$$

This asymptotic expression for η_0 and that for $f(\infty)$ have been compared with exact numerical calculations (table 2). As expected the agreement is closest for smaller values of R.

R	η_0 Numerical	η_0 Asymptotic	$f(\infty)$ Numerical	$f(\infty)$ Asymptotic
0·1	1·687	1·678	−0·212	−0·188
0·05	1·6014	1·5966	−0·1418	−0·1352
0·01	1·529655	1·529278	−0·053065	−0·052757
0·001	1·534945	1·534940	−0·011943	−0·011939
0·0001	1·551750	1·5517498	−0·002599	−0·0025988

TABLE 2

5. Discussion of results

The frictional moment on the part of the disk with radius a is

$$M = -2\pi\bar{\rho}\nu\int_0^a r^3\frac{\partial v}{\partial z}(z=0)\,dr = -\tfrac{1}{2}\pi\bar{\rho}\nu\,\frac{\Omega^2}{w_0}a^4 g'(0), \tag{48}$$

where the value of $g'(0)$ can be obtained from the solution developed in the injection layer. Three terms of the expansion (two of which are zero) can be obtained from (11), (13) and (15). Watson (1966) in dealing with a similar problem, has shown that one need not integrate the equations if one is interested in the flow field in the immediate neighbourhood of the disk only. Indeed, the problem posed in the injection layer is an initial-value problem with boundary conditions at the disk and thus double series expansions involving integral powers of η and R can be used. Substitution in the equations leads to recursive formulae for the coefficients of these series. It is a simple matter to obtain an algorithm that generates these coefficients. In this way the following expression is found for the frictional moment

$$M = \pi\bar{\rho}a^4\nu^2\frac{\Omega^3}{w_0^3}[1-22R^2+892R^4-44856R^6+2529584\,R^8+\ldots]. \tag{49}$$

The calculation of the coefficients was carried as far as possible until significance was lost because of the limitations of the computer. By calculating the ratio of two successive coefficients it became clear that the series expansion of (49) converges for

$$R^2 < \tfrac{1}{66}, \tag{50}$$

i.e. the solution presented for the injection layer is valid at the disk for $R < 0{\cdot}12$. The results were checked by comparison with very accurate numerical integrations of the complete equations. This showed that by using the series expression of (49) until $O(R^{14})$, at least eight significant decimals can be obtained if $R < 0{\cdot}05$. For $R = 0{\cdot}1$ the numerical integrations still confirmed the three significant decimals of the series.

These results show that for $R < 0{\cdot}12$ the influence of the viscous boundary layer is trancendentally small at the wall, since (49) is obtained without any knowledge of the viscous boundary layer. For $R > 0{\cdot}12$ (49) is no longer valid whence we must assume that M is influenced then by the viscous sublayer. It seems to follow that for $R \sim 0{\cdot}12$ the viscous sublayer detaches from the wall.

Another fact which deserves some attention is the rather peculiar dependence of η_0 on R. Numerical integrations show that, for R large enough, η_0 will be larger than $\frac{1}{2}\pi$ and $\eta_0(R)$ is a monotonically increasing function of R. η_0 reaches a minimum value, which is smaller than $\frac{1}{2}\pi$ at about $R = 0{\cdot}01$. For yet smaller values of R it will increase until it reaches the value $\frac{1}{2}\pi$ for $R = 0$. Since η_0 is related to the thickness of the injection layer

$$z_0 = (w_0/\Omega)\,\eta_0(R), \tag{51}$$

it is possible to study the dependence of z_0 on the viscosity ν. Thus for the larger values of R it is seen that increasing the viscosity will lead to increasing values of z_0. This is indeed an expected behaviour since the centrifugal force will be opposed stronger for larger values of the viscosity. Thus the fluid will spiral out slower and hence the injection layer will be thicker.

For R close to 0 an opposite behaviour is encountered: if R is increased from 0 to a slightly larger value, the injection layer will become thinner. In explaining this we should observe that for $R = 0$ the velocity gradients, in particular $\partial u/\partial z$, are discontinuous at $\eta = \frac{1}{2}\pi$. For $\eta = \frac{1}{2}\pi - 0$ we have $\partial u/\partial z = -\Omega^2 r/w_0$ whereas $\partial u/\partial z = 0$ for $\eta = \frac{1}{2}\pi + 0$. The velocity u itself is equal to zero at $\eta = \frac{1}{2}\pi$. If slight viscosity effects are introduced the velocity gradient has an accelerating effect on layers of lowest velocity and thus the fluid in the neighbourhood of $\eta = \frac{1}{2}\pi$ will move radially outwards. It should be noted that this is a purely viscous effect, i.e. no centrifugal forces are causing this radial outflow. Indeed, as we have seen, centrifugal forces do not affect the zero-order term of the viscous sublayer. These remarks are substantiated if we consider the two-dimensional problem where a shear layer with $u = -Ky$ $(y \leqslant 0)$ comes into contact at $x = 0$ with a region of quiescent fluid. Here u is the velocity in the x direction. If the flow is inviscid both regions will not influence each other and the separating plane will be at $y = 0$. If the fluid is viscous a boundary layer will develop about $y = 0$. Upon substituting

$$\tilde{\psi} = -\left(\frac{9K\nu^2x^2}{8}\right)^{\frac{1}{3}} Q(s); \quad s = y\left(\frac{K}{3\nu x}\right)^{\frac{1}{3}} \tag{52}$$

in the two-dimensional boundary-layer equations we obtain

$$Q''' - QQ'' + \tfrac{1}{2}(Q')^2 = 0. \tag{53}$$

and the boundary conditions

$$Q'(\infty) = 0; \quad \lim_{s \to -\infty} (Q' - 2s) = 0. \tag{54}$$

But this system is the same as that which governs the main term of the viscous sublayer above the rotating disk. And thus the separating surface will be

$$y_0 = -0{\cdot}430187\left(\frac{3\nu x}{K}\right)^{\frac{1}{3}}. \tag{55}$$

Equation (53) is recognized as that used by Goldstein, with boundary conditions different from those used here, to describe the near wake of a flat plate (Goldstein 1930).

The analogy with our problem for the rotating disk becomes complete if we consider a plane jet issuing into a quiescent fluid at $x = 0$, $-l \leqslant y \leqslant l$. For our purpose the exact velocity distribution in the jet does not matter. We only require that at the edges of the slit the velocity gradient has a finite jump while the velocity itself is equal to zero at these edges. It is well known that such a problem can be solved by local series expansion about the planes $y = \pm l$ about which there are viscous boundary layers. The above analysis shows that the injected layer will become thinner at first. Due to the finiteness of the jet this analysis is valid for small values of x only. It is well known that the jet will eventually become a Bickley–Schlichting jet which spreads out far beyond the lateral dimensions of the slit.

As a last subject for discussion we will consider the influence of blowing on the uniform inflow far above the disk. From (1), (2) and (21) it follows immediately using table 1 that for $R \ll 1$ and $z \gg w_0/\Omega(\frac{1}{2}\pi)$

$$w \sim -1{\cdot}210\left(\frac{\nu^2\Omega^2}{w_0}\right)^{\frac{1}{3}}; \tag{56}$$

so that the inflow decreases to zero as $w_0 \to \infty$, independently of ν and Ω. Since inflow exists in order to feed a viscous boundary layer (no inflow is encountered for $R = 0$) this shows that the detached boundary layer is fundamentally different from the boundary layer that is attached. In view of this it is interesting to compare (56) with the inflow in the non-blowing case (Dorfman 1963)

$$w \sim -0{\cdot}886(\nu\Omega)^{\frac{1}{2}}. \tag{57}$$

Thus, with decreasing ν and Ω the inflow in the case of blowing decreases more rapidly than that of the non-blowing case.

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